Minima among subsets of random variables, Random permutations, and Load-sharing survival models

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Let $(X_1, .., X_m)$ be a vector of, generally interdependent, non-negative random variables satisfying the *no-tie* condition $\mathbb{P}(X_i = X_j) = 0, \forall i \neq j$. Denoting by $X_{1:m}, ..., X_{m:m}$ the corresponding order statistics, we consider both the $\{1, .., m\}$-valued random variables $J_1, ..., J_m$ defined by

$$J_h = i \Leftrightarrow X_{h:m} = X_i$$

and the family $\mathcal{A} \equiv \{\alpha_j(A); A \subseteq \{1, .., m\}, j \in A\}$, where the numbers $\alpha_j(A)$ are defined by

$$\alpha_j(A) := \mathbb{P}(\min_{i \in A} X_i = X_j).$$

and are determined by $\boldsymbol{P_J}$, the joint probability distribution of $\boldsymbol{J} \equiv (J_1, ..., J_m)$.

Starting from a multivariate extension of the concept of *stochastic precedence,* in the first part of this talk I will illustrate the notion of *ranking pattern concordant with* $(X_1, .., X_m)$ (see [1]). For any subset $A \subseteq \{1, .., m\}$ the latter object, denoted by the symbol $\boldsymbol{\sigma}$, describes the ranking among the element of A, induced by the numbers $\alpha_j(A)$.

Motivating examples arising from different fields will be given and the possible appearance of paradoxical outcomes will be pointed out.

The second part of the talk will be devoted to showing that, in an appropriate sense, any arbitrary "paradox" in the family $\mathcal{A}$ can actually be produced. Such a problem will be reduced to proving, for an arbitrary ranking pattern $\boldsymbol{\sigma}$, the existence of suitable probability distributions $\boldsymbol{P_J}$ over the permutations of $1, 2, ..., m$. More precisely it will be shown that a special type of load-sharing model can be explicitly constructed under which $\boldsymbol{\sigma}$ is concordant with $(X_1, .., X_m)$.

A further result goes as follows: for an arbitrary probability distribution ρ over the space of the permutations of $1, ..., m$, a load-sharing model can be constructed under which the distribution $\boldsymbol{P_J}$ does coincide with ρ.

The talk will conclude with a discussion about the logical differences between the two results and about respective applications to the concept of system signature in the field of systems' reliability and to the analysis of paradoxes arising in a standard scenario of voting theory with m candidates.

These arguments are based on joint work with Emilio De Santis at Sapienza University, Rome.

[1] E. De Santis, F. Spizzichino. Construction of aggregation paradoxes through Load-sharing models. *Adv. Appl. Probab.*, to appear.

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